

## Two-Dimensional Co-Oscillations in a Rectangular Bay: Possible Application to Water-Level Problems

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*The two-dimensional response of a rectangular bay of uniform depth to a fluctuating water-level disturbance imposed at its mouth is examined in the framework of linear shallow-water equations on a nonrotating earth. The imposed forcing is periodic in time but spatially varying in the transverse direction along the mouth of the bay. The response is presented both in terms of the amplification factor, which is the ratio of the imposed amplitude at the mouth to that at the closed end of the bay, and the structure of the height field within the bay. The two-dimensional character of the response becomes more pronounced as the wavelength of the disturbance at the mouth decreases and as the width of the bay increases. Positive and negative amphidromic systems can be generated in the bay for disturbances propagating along the mouth of the bay even though the earth's rotation is neglected. The origin of the water-level fluctuations at the mouth of the bay could be due to tides, storm surges, or tsunamis. This study indicates the importance of measuring the spacial variations in the water-level fluctuations along the mouth of the bay, instead of assuming them to be spatially uniform, when attempting to explain the water-level response within the bay.*

**Keywords** co-oscillations, rectangular bay, two-dimensional problem, no-rotation case

All bays are subjected to forcing at their mouths due to water-level fluctuations imposed by the external water bodies to which they are connected. These water-level fluctuations in the external water bodies could be produced by any number of causes, such as astronomical tides, storm surges, or tsunamis. This imposed forcing at the mouth produces a co-oscillating response within the bay. These co-oscillations in a rectangular bay have been previously examined in the context of one-dimensional shallow-water theory for simple-analytical-shaped bottom topographies. For basins of variable shapes and bottom

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topographies, the problem has been examined in the context of the channel equations using numerical procedures (see Defant, 1961). In these one-dimensional studies, the water-level fluctuation at the mouth is periodic in time and uniform in space along the mouth of the bay. The response of the bay is generally given in terms of the amplification factor, which is the ratio of the imposed forcing elevation at the mouth to that at the closed end of the bay. This amplification factor depends on the basin geometry, bottom topography, and the frequency of the forcing at the mouth. For a rectangular bay of length  $L$  and uniform depth  $H$ , for example, the amplification factor is given by  $1/\cos(\omega L/C)$ , where  $C \equiv \sqrt{gH}$  is the speed of long gravity waves and  $\omega$  is the frequency of forcing imposed at the mouth. The amplification factor is generally  $\geq 1$ , since  $\cos(\omega L/C)$  is always  $\leq 1$ , indicating that the amplitude at the head of the bay is larger than the forcing amplitude imposed at the mouth. The amplification becomes infinite (resonant response) in the inviscid case whenever  $\omega L/C$  becomes an odd multiple of  $\pi/2$ , which represents the value of the frequency of a free oscillation mode of the bay.

These simple considerations are made complicated in nature due to various factors. One such factor is that all free modes of oscillation of the bay are assumed to be characterized by the existence of a nodal line—a line of zero water-level amplitude—at the mouth of the bay. Imposition of this condition makes the frequencies of free oscillations equal to odd multiples of  $\pi/2$  in the one-dimensional case mentioned above. In reality, the nodal lines for different free modes of a bay connected to a larger water body occur at different locations within the domain of the entire system as shown, for example, in the study of Rao et al. (1976) on Lake Michigan. Hence, some of the free modes of a bay can have a nonzero water-level amplitude at what might normally be considered the physical location of the mouth of the bay. This feature then changes the frequency of the mode from what it would have been if a nodal line were imposed at the mouth. The resulting effect is a change in the amplification of a forced oscillation (and conditions for resonance within the bay). This problem has been addressed by Heaps (1975).

Another factor that changes the simple one-dimensional response is that the amplitude of the forcing imposed at the mouth will not normally be uniform along the mouth of the bay, even for a narrow bay, but will exhibit some spacial variation. Such a spacial variation in the forcing will result in a two-dimensional response within the bay. As a consequence, the magnification factors at the head of the bay will be not only different from what one would obtain for a spatially uniform forcing at the mouth, but would also change in the transverse direction. In the study of the oscillations of Green Bay by Heaps et al. (1982), several discrepancies were encountered between the observed and computed water levels. As speculated by them, some of these differences could indeed be related to the fact that spacial variations in the forcing imposed by the Lake Michigan water level at the mouth of the Green Bay have not been taken into account. In comments on a study of Dorresteijn (1983), Murty and El-Sabh (1989) speculated that local topography and friction may be responsible for producing anomalous clockwise (negative)-rotating amphidromic systems in the co-oscillating tides in some of the marginal seas in the northern hemisphere and vice versa in the southern hemisphere. This is probably true for large marginal seas and bays, whose dynamics are influenced significantly by the earth's rotation. As will be shown in this study, for smaller water bodies or those at very low latitudes, for which the effects of the earth's rotation are negligible, disturbances propagating along the mouth can produce co-tidal oscillations that exhibit either positive or negative amphidromic systems even though the earth's rotation is ignored.

In order to illustrate the nature of the two-dimensional response of a bay to a periodic but spatially varying water-level fluctuation imposed at the mouth, we consider here a

simple case of a nonrotating rectangular bay of uniform depth. Modifications due to the possible nonexistence of a nodal line at the mouth for some of the free modes, effects of nonuniform bottom topography, and the earth's rotation will be considered later.

### Basic Equations and Solution

The two-dimensional linear shallow-water equations on a nonrotating earth are

$$\begin{aligned} \frac{\partial u}{\partial t} &= -g \frac{\partial \eta}{\partial x} & \frac{\partial v}{\partial t} &= -g \frac{\partial \eta}{\partial y} \\ \frac{\partial \eta}{\partial t} &+ \left( \frac{\partial u H}{\partial x} + \frac{\partial v H}{\partial y} \right) & &= 0 \end{aligned} \quad (1)$$

In these equations,  $u$  and  $v$  are the horizontal velocities in the  $x$  and  $y$  directions,  $t$  is time,  $\eta$  is the water-level fluctuation about the mean,  $H$  is the depth of the water in the undistributed state, and  $g$  is the gravitational acceleration. We assume that the bay occupies the domain  $0 \leq x \leq L$  (length of the bay) in the longitudinal direction and  $0 \leq y \leq B$  (breadth of the bay) in the transverse direction. The bay is closed at  $x = L$  (head of the bay) and along  $y = 0$  and  $B$ . It is open to a large external water body along its mouth at  $x = 0$ , where it is connected to a larger water body (see Figure 1). The appropriate boundary conditions for the free oscillations are that

$$u = 0 \text{ at } x = L \text{ (head of the bay)} \quad (2a)$$

$$v = 0 \text{ at } y = 0 \text{ and } B \quad (2b)$$

$$\eta = 0 \text{ at } x = 0 \text{ (mouth of the bay)} \quad (2c)$$

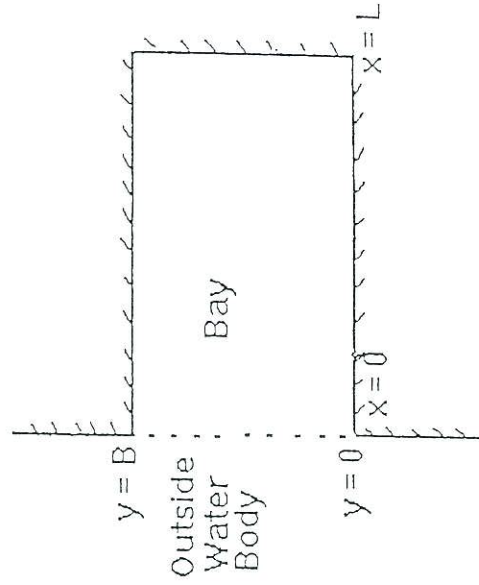


Figure 1. Geometry of the bay.



Eliminating  $u$  and  $v$  in Eq. (1) results in a single equation for  $\eta$ ,

$$\frac{\partial^2 \eta}{\partial t^2} = \left( \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} \right) \tag{3}$$

In the above equation, all quantities are made nondimensional using the length of the bay  $L$  for  $x$  and  $y$  coordinates,  $L/C$  for time, and a scale amplitude  $\eta_0$  for the height field. In these nondimensional units, the domain of the bay is  $0 \leq x \leq 1$  in the longitudinal direction and  $0 \leq y \leq \epsilon$  ( $\equiv BL$ ) in the transverse direction.

Assuming a periodic time dependence of the form  $\cos(\sigma t)$ , where  $\sigma$  is the nondimensional frequency of free oscillations, the solution for  $\eta$  that satisfies the boundary conditions given in Eq. (2) is

$$\eta = \sin \left[ (2m + 1) \frac{\pi x}{2} \right] \cos \left( \frac{n\pi y}{\epsilon} \right) \cos(\sigma t) \tag{4}$$

The free oscillation frequencies are given by

$$\sigma = \pi \left[ \frac{(2m + 1)^2}{4} + \frac{n^2}{\epsilon^2} \right]^{1/2} \tag{5}$$

in which  $m$  and  $n$  are integers that can take on the values  $0, 1, 2, 3, \dots$ . All the free modes that correspond to  $n = 0$  are obviously the one-dimensional longitudinal oscillations, and the frequencies of all these modes are equal to odd multiples of  $\pi/2$ . The mode that corresponds to  $m = 0$  and  $n = 0$  is the lowest Helmholtz mode.

For the forced oscillations, the governing equation is still the same as in Eq. (3), but the boundary condition at the mouth for the fluctuation of the water level is now given by

$$\eta(y) = f(y) \cos \omega t + g(y) \sin \omega t \quad \text{at } x = 0 \tag{6}$$

instead of Eq. (2c), which holds for free oscillation modes. In Eq. (6),  $\omega$  is the frequency of the forcing from the larger water body to which the bay is connected. The functions  $f(y)$  and  $g(y)$  represent the distribution of the forcing amplitude along the mouth of the bay. The solution to the forced problem is given by

$$\begin{aligned} \eta(x, y, t) = & \sum_n F_n \cos \alpha_n \omega (x - 1) \cos \frac{n\pi y}{\epsilon} \cos \omega t \\ & + \sum_n G_n \cos \alpha_n \omega (x - 1) \cos \frac{n\pi y}{\epsilon} \sin \omega t \end{aligned} \tag{7}$$

where the coefficient  $\alpha_n$  is given by

$$\alpha_n = \left( 1 - \frac{n^2 \pi^2}{\epsilon^2 \omega^2} \right)^{1/2} \tag{8}$$

The expansion coefficients in Eq. (7) are determined through the condition that

$$\left. \begin{aligned} \sum_n F_n \cos \alpha_n \omega \cos \frac{n\pi y}{\epsilon} &= f(y) \\ \sum_n G_n \cos \alpha_n \omega \cos \frac{n\pi y}{\epsilon} &= g(y) \end{aligned} \right\} \text{at } x = 0 \tag{9}$$

Hence, the expansion coefficients are given by

$$\begin{aligned} F_n \cos \alpha_n \omega &= \frac{\gamma_n}{\epsilon} \int_0^\epsilon f(y) \cos \frac{n\pi y}{\epsilon} dy \\ G_n \cos \alpha_n \omega &= \frac{\gamma_n}{\epsilon} \int_0^\epsilon g(y) \cos \frac{n\pi y}{\epsilon} dy \end{aligned} \tag{10}$$

where

$$\begin{aligned} \gamma_n &= 1 & \text{for } n = 0 \\ \gamma_n &= 2 & \text{for } n \neq 0 \end{aligned}$$

The coefficient  $\alpha_n$  may be real or imaginary depending on the values of  $n$ ,  $\epsilon$ , and  $\omega$ .

### Results for Specific Forcing Functions

The forcing imposed at the mouth of the bay depends on its orientation with respect to the connecting water body and the dynamics of the coupled system. In the context of nonrotating dynamics, this forcing can be of the nature of a standing wave at the mouth of the bay pumping the water level in a periodic nature or it can be of the nature of a wave propagating tangentially at the entrance of the bay. In order to take care of both of these possibilities, we shall consider forcing functions as defined below:

$$f(y) = \cos ky \quad \text{and} \quad g(y) = 0 \tag{11a}$$

$$f(y) = 0 \quad \text{and} \quad g(y) = \sin ky \tag{11b}$$

$$f(y) = \cos ky \quad \text{and} \quad g(y) = \sin ky \tag{11c}$$

The forcing functions given in Eqs. (11a) and (11b) represent standing waves at the mouth of the bay, whereas the forcing from Eq. (11c) represents a disturbance propagating along the mouth of the bay with a phase speed given by  $k/\omega$ . The expansion coefficients in each case are given by

$$F_n \cos \alpha_n \omega = \frac{\gamma_n k \epsilon (-)^n \sin k \epsilon}{k^2 \epsilon^2 - n^2 \pi^2} \tag{12a}$$

for the cosine forcing (11a), and

$$G_n \cos \alpha_n \omega = \frac{\gamma_n k \epsilon [1 - (-)^n \cos k \epsilon]}{k^2 \epsilon^2 - n^2 \pi^2} \tag{12b}$$