The Time Dependent Ray Method for Calculation of Wave Transformation on Water of Varying Depth and Current [1]

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Abstract

This paper focuses on an aspect of the numerical calculation of wave refraction using the ray tracing or the semi-Lagrangian approach. The conventional ray method for steady state depth and current fields is extended to the unsteady situation. It is shown that a factor which characterizes the unsteadiness of the medium can be included explicitly in the calculation of the wave energy amplification. The contribution associated with this factor on wave energy is $c_o^2/(C_g + U)^2$, where C_g and U are the wave group velocity and current velocity, respectively, and c_o is the phase velocity of the wave-like fluctuation of currents and depth (e.g., caused by tides) in the direction of the ray path, i.e., in the direction of wave energy propagation.

Introduction

Efforts to include the effects of wave refraction due to spatially and/or temporally varied current and water depth in a numerical wave prediction model have been made by a number of researchers (see, e.g., Collins, 1972; Cavaleri and Rizzoli, 1981; Chen and Wang, 1983; Tolman, 1989,1991; Hubbert and Wolf, 1991). Numerical schemes for wave propagation and refraction calculations employed by these researchers can be conveniently classified into two types of approach. The Eulerian approach applies a finite difference scheme to obtain solutions at each grid point over the area of interest simultaneously. The Lagrangian approach uses the ray tracing technique to derive solutions at each specified point independently.

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The advantage of the Eulerian approach is that it provides a synoptic view of the wave pattern involving all frequency and directional components over the entire area and allows addition of the various source and sink terms including nonlinear wave-wave interactions. However, finite difference schemes for the convective transport equation have a common problem - the solution always involves unavoidable numerical errors (numerical dispersion and diffusion). As a result, the physical significance of their solutions is obscured.

In contrast, the Lagrangian approach (or what comes to the same thing: the method of characteristics or the ray method) follows a wave packet along the wave ray, thus avoiding the troublesome convective term and providing a strong physical realization of wave propagation. It provides a true path and time for any choice of wave parameters arriving at a specific point. Consequently, the loss of spatial generality is compensated by an increase in temporal accuracy.

There are limitations to this approach. It does not provide the synoptic view of the wave field over the entire area of interest at each computational time step. It is incapable of modeling the effects of source terms which involve all spectral components at each time step, such as nonlinear wave-wave interactions and white capping. It is also difficult to specify quantitatively the energy amplification factor in the vicinity of ray crossings and caustics which may occur if waves propagate over a complicated bottom topography or current field.

An early effort to combine the strengths and eliminate the shortcomings of both the finite difference and ray tracing techniques has been made by Barnett et al. (1969, also see, Allender et al.,1985). They developed a scheme which constructed a set of parallel rays over the ocean for each specified wave direction. Along each ray, the distance between ray points is determined according to the group velocity of the given frequency and time interval. The wave energy, considered as particles, hop from ray point to ray point at appropriate multiples of the time step. The wind field is provided on a net of grid points defined over the area of interest. At each given time step, the wave spectrum at any grid point is accumulated from the nearest ray point for each frequency and direction. After the spectrum is constructed, the processes of wave growth, dissipation and wave-wave energy transfer are executed. The spectral components are then re-distributed back to the appropriate ray points. Their approach can be applied only in deep water without currents and under steady state conditions since it assumes the constancy of wave direction along the rays.

It is of interest to note that the so-called semi-Lagrangian technique, which has gained widespread recognition as an efficient way to integrate the primitive equations of numerical weather prediction, also has been applied to the numerical prediction of ocean waves in recent years. Basically, it is an extension of the classical treatment of the refraction problem at a given site using the backward ray tracing technique (Dorrestein, 1960). With this approach, each grid point of a grid mesh covered the entire area of concern is considered to be a target point to receive the wave energy of

all spectral components in a specified time step from various source locations. The source location of each component (the departure point) is determined by the method of ray back-tracing for the given time step. The localized physical processes such as wave growth by wind, dissipation due to whitecapping and wave-wave energy transfer are calculated at each grid point. The calculated spectral energy densities at all grid points serve to provide information for deriving the spectral energy density at each source location. Since it is impossible to have the departure point always coincide with a certain grid point, an appropriate interpolation procedure is performed to obtain the energy content at the departure point based on the values at neighboring grid points. A feasibility study of this approach along with different methods for spatial interpolation as a part of this system, has been studied by Ryabinin (1991).

The scheme is potentially attractive because of its high accuracy and efficiency in computation by allowing large time intervals and low spectral resolutions. In shallow water, the energy dissipation due to bottom friction can be computed along the rays. The effect of sub-grid irregularity in bottom topography and/or surface currents on the wave energy amplification can be calculated explicitly, and the problem of crossing rays and caustics can be avoided as the wave propagation distance is relatively short in a given time step. In addition, the underlying grid mesh can be arranged irregularly depending on the accuracy of the input data and the requirements for wave information in the area of interest.

A problem in applying this semi-Lagrangian approach, however, remains to be solved. In connection with the study of wave-current interaction problem in the southern North Sea, Tolman (1990) pointed out that for the large scale continental shelf unsteadiness of current and water depth induced by tides must be considered if the time scale of variation in current and depth (typically 12 hours) is not large compared to the travel time of the waves through the area of interest. The wave energy amplification factor derived from the conventional ray method is derived based on the assumption of steady state water depth and flow conditions. Therefore the effect of unsteadiness in depth and currents cannot be evaluated. In this paper, the time dependent ray method is derived to express explicitly factors involved in calculating the change of wave spectrum due to the existence of unsteady and irrelegular depth and current fields.

Basic Equations

The change of wave field due to the presence of varying currents and bathymetry can be specified based on the dynamic conservation of wave action and the kinematic conservation of wave number or wave crests along characteristic curves or rays (Bretherton and Garrett, 1969; Phillips, 1977). The path of a ray is determined by simultaneous solution of the following set of equations:

$$\frac{dx_j}{dt} = \frac{\partial \omega}{\partial k_j} = c_{g_j} + u_j, \tag{1}$$

$$\frac{dk_i}{dt} = -\frac{\partial\omega}{\partial\lambda}\frac{\partial\lambda}{\partial x_i} = -\frac{\partial\sigma}{\partial h}\frac{\partial h}{\partial x_i} - k_j\frac{\partial u_j}{\partial x_i}, \tag{2}$$

and

$$\frac{d\omega}{dt} = \frac{\partial\omega}{\partial\lambda}\frac{\partial\lambda}{\partial t} = \frac{\partial\sigma}{\partial h}\frac{\partial h}{\partial t} + k_i\frac{\partial u_i}{\partial t}, \tag{3}$$

where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \frac{\partial\omega}{\partial k_j}\frac{\partial}{\partial x_j}. \tag{4}$$

Here the wave-number vector $\mathbf{k} = (k_1, k_2)$, group velocity $\mathbf{C_g} = (c_{g_1}, c_{g_2})$, flow velocity $\mathbf{U} = (u_1, u_2)$, and the horizontal cartesian coordinates $\mathbf{x} = (x_1, x_2)$. The apparent frequency, ω, is given by $\omega(\mathbf{k}, \lambda) = \sigma + \mathbf{k}\cdot\mathbf{U}$, where σ, the intrinsic frequency of waves in a frame of reference moving with flow velocity $\mathbf{U}(\mathbf{x}, t)$, obeys the dispersion relationship, $\sigma = (gk\tanh kh)^{1/2}$, where g is the gravitaional acceleration, $k = \mid \mathbf{k} \mid$, and $h(\mathbf{x}, t)$ is water depth. $\lambda(\mathbf{x}, t)$ represents local properties of the medium, i.e., h and $\mathbf{U}$. Equation 3 indicates that if water depth and current velocity do not vary with time, ω remains constant along the rays.

Conservation of wave action for a slowly varying wavetrain of small amplitude can be expressed in terms of rays as

$$\frac{d}{dt}\left(\frac{E}{\sigma}\right) + \left(\frac{E}{\sigma}\right)\nabla\cdot(\mathbf{C_g} + \mathbf{U}) = 0. \tag{5}$$

E is the local wave energy per unit area (proportional to the square of the wave amplitude). The wave action is defined as E/σ. For a continuous spectrum, the energy density of a group of waves whose wave-numbers lie in the element of area δA of the wave-number plane, specified by the vectors $\mathbf{k}$, $\mathbf{k} + \delta\mathbf{k}'$, and $\mathbf{k} + \delta\mathbf{k}''$ is given by

$$\delta E(\mathbf{k}) = \rho g F(\mathbf{k})\,\delta A, \tag{6}$$

in which

$$\delta A = |\delta\mathbf{k}' \times \delta\mathbf{k}''|. \tag{7}$$

$F(\mathbf{k})$ is the spectral density and ρ the water density. By applying the kinematic conservation principle, Phillips (1977) has shown that

$$\frac{d}{dt}\,\delta A + \delta A\,\nabla\cdot(\mathbf{C_g} + \mathbf{U}) = 0. \tag{8}$$

Therefore by substituting eqs.(6) and (8) into eq.(5) we have

$$\frac{d}{dt}\left(\frac{F(\mathbf{k})}{\sigma}\right) = 0. \tag{9}$$

Equation 9 expresses the conservation of spectral wave action density along the ray. In the absence of a current, $F(\mathbf{k})$ remains constant along the ray. This result was first demonstrated by Longuet-Higgins (1957).

Time Dependent Ray Method

Equation 8 cannot be directly integrated along a ray because knowledge of neighboring solutions is required to determine the divergence of velocity. Further complications arise if variations of currents and/or water depth with time cannot be ignored. Under these conditions, ω and $\mathbf{C_g} + \mathbf{U}$ are not independent of time. In order to solve this problem in a general manner without making a usual assumption that ω is independent of time, the approach of Shen and Keller (1975) is employed. We begin by defining

$$\frac{dt}{d\gamma_1} = \mu. \tag{10}$$

Here $\mu(t, x, y)$ is an arbitrary but non-zero proportionality function. The choice of μ determines the nature of the parameter γ_1 as we shall see. Equations 1, 2, 3 and 8 consequently can be expressed, respectively, as

$$\frac{dx_j}{d\gamma_1} = \mu(c_{g_j} + u_j), \tag{11}$$

$$\frac{dk_i}{d\gamma_1} = -\mu\left(\frac{\partial\sigma}{\partial h}\frac{\partial h}{\partial x_i} - k_j\frac{\partial u_j}{\partial x_i}\right), \tag{12}$$

$$\frac{d\omega}{d\gamma_1} = \mu\left(\frac{\partial\sigma}{\partial h}\frac{\partial h}{\partial t} + k_i\frac{\partial u_i}{\partial t}\right), \tag{13}$$

and

$$\mu^{-1}\frac{d}{d\gamma_1}\,\delta A + \delta A\,\nabla\cdot(\mathbf{C_g} + \mathbf{U}) = 0, \tag{14}$$

where

$$\frac{d}{d\gamma_1} \equiv \mu\left\{\frac{\partial}{\partial t} + \frac{\partial\omega}{\partial k_j}\frac{\partial}{\partial x_j}\right\}. \tag{15}$$

We now introduce the Jacobian

$$J(\gamma_1, \gamma_2, \gamma_3) = \frac{\partial(x_1, x_2, x_3)}{\partial(\gamma_1, \gamma_2, \gamma_3)} = det\left(\frac{\partial x_i}{\partial\gamma_j}\right), \qquad i, j = 1, 2, 3. \tag{16}$$

Here, for convenience, we have set $\mathbf{r} = (\mathbf{x}, c_o t) = (x_1, x_2, x_3)$, where c_o is a constant reference speed associated with the ambient medium, and $(\gamma_1, \gamma_2, \gamma_3)$, is a set of parameters describing a point on a particular ray. γ_1 is the running parameter which varies along the ray and γ_2 and γ_3 are labeling parameters specifying a particular ray and are constants on each ray of the family of rays. In terms of these parameters, a point on a ray of a family can be respresented by $\mathbf{r} = \mathbf{r}(\gamma_1, \gamma_2, \gamma_3)$. For fixed values of γ_2 and γ_3, this represents the equation of a ray.

We observe that the determinant can be expanded in terms of the cofactors such that

$$\sum_{j=1}^{3}\frac{\partial x_m}{\partial\gamma_j}cof\frac{\partial x_i}{\partial\gamma_j} = J\delta_{im}. \tag{17}$$

Here δ_{im} is the Kronecker symbol and cof denotes the cofactor. If $i = m$, $\delta = 1$ and equation 17 follows from the rule for the expansion of a determinant by cofactors. If $i \neq m$, it follows from the fact that a determinant with two identical rows vanishes. We now differentiate equation 16 with respect to γ_1, observing that the derivative of a determinant is the sum of the derivatives of all its elements, each multiplied by its cofactor. Then we write $\partial/\partial\gamma_j = \sum_{m=1}^{3}(\partial x_m/\partial\gamma_j)\partial/\partial x_m$ and obtain

$$\begin{aligned}\frac{\partial J}{\partial\gamma_1} &= \sum_{i,j=1}^{3}\frac{\partial^2 x_i}{\partial\gamma_j\partial\gamma_1}cof\frac{\partial x_i}{\partial\gamma_j} = \sum_{i,j=1}^{3}\frac{\partial}{\partial\gamma_j}\left(\frac{\partial x_i}{\partial\gamma_1}\right)cof\frac{\partial x_i}{\partial\gamma_j}\\ &= \sum_{i,j,m=1}^{3}\frac{\partial x_m}{\partial\gamma_j}\frac{\partial}{\partial x_m}\left(\frac{\partial x_i}{\partial\gamma_1}\right)cof\frac{\partial x_i}{\partial\gamma_j} = \sum_{i,m=1}^{3}J\delta_{im}\frac{\partial}{\partial x_m}\left(\frac{\partial x_i}{\partial\gamma_1}\right)\\ &= J\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(\frac{\partial x_i}{\partial\gamma_1}\right) \qquad (18)\end{aligned}$$

Here we have used equation 17 in equation 18 and evaluate the sum over j and then the sum over m. Finally we use equations 10 and 11 and find

$$\begin{aligned}\frac{\partial J}{\partial\gamma_1} &= J\left\{\frac{\partial}{\partial t}\left(\frac{dt}{d\gamma}\right)+\frac{\partial}{\partial x}\left(\frac{dx}{d\gamma}\right)+\frac{\partial}{\partial y}\left(\frac{dy}{d\gamma}\right)\right\}\\ &= J\left\{\frac{\partial\mu}{\partial t}+\nabla\cdot\mu(\mathbf{C_g}+\mathbf{U})\right\}\\ &= J\left\{\frac{d\mu}{dt}+\mu\nabla\cdot(\mathbf{C_g}+\mathbf{U})\right\}. \qquad (19)\end{aligned}$$

By using $d\mu/dt = (d\mu/d\gamma)(d\gamma/dt) = (d\mu/d\gamma)\mu^{-1}$ in equation 19 and rearranging the equation, we have

$$\nabla\cdot(\mathbf{C_g}+\mathbf{U}) = \frac{1}{\mu}\left\{\frac{1}{J}\frac{dJ}{d\gamma}-\frac{1}{\mu}\frac{d\mu}{d\gamma}\right\}. \qquad (20)$$

Therefore equation 14 becomes

$$\frac{d}{d\gamma_1}\ln(\frac{J\delta A}{\mu}) = 0, \qquad (21)$$

which states that the quantity $J\delta A/\mu$ is conserved along the space-time ray. Since equation 5 has the same form as equation 8 we can conclude that

$$\frac{JF\delta A}{\mu\sigma} = constant \qquad (22)$$

along the space-time ray.

Interpretation of $J(\gamma)$

The Jacobian J in equation 22 can be given a geometric interpretation. We envisage a curved surface $f(x,y,t)$ formed by a rectifiable wave action front as it moves forward during time δt. A point on this surface $\mathbf{x}$ can be identified by the running parameter γ_1 and the labeling parameters γ_2 and γ_3. The parameter γ_1 gives the location of the point on a ray whose initial position on a wave front is given by parameter γ_2. This action front is specified by γ_3 related to some initial time. This wave action front resembles the so-called initial manifold in the context of solving some initial value problems.

We note that the Jacobian defined by equation 16 can also be expressed as a triple scalar product

$$J = \frac{d\mathbf{r}}{d\gamma_1}\cdot\left(\frac{\partial\mathbf{r}}{\partial\gamma_2}\times\frac{\partial\mathbf{r}}{\partial\gamma_3}\right) = \frac{\partial\mathbf{r}}{\partial\gamma_3}\cdot\left(\frac{d\mathbf{r}}{d\gamma_1}\times\frac{\partial\mathbf{r}}{\partial\gamma_2}\right). \qquad (23)$$

Since

$$\left|\frac{d\mathbf{r}}{d\gamma_1}\right|^2 = \left(c_o\frac{dt}{d\gamma_1}\right)^2+\left(\frac{dx}{d\gamma_1}\right)^2+\left(\frac{dy}{d\gamma_1}\right)^2 = \mu^2[c_o^2+(\mathbf{C_g}+\mathbf{U})^2], \qquad (24)$$

if we choose

$$\mu = [c_o^2+(\mathbf{C_g}+\mathbf{U})^2]^{-1/2}, \qquad (25)$$

then it follows from equation 24 that $(d\mathbf{r}/d\gamma)^2 = 1$. Therefore, $d\mathbf{r}/d\gamma_1 = \mathbf{s}$ is the unit tangent vector to a ray, and the parameter γ_1 is the arclength of the space-time ray. Since we are concerned with the change of wave characteristics associated with this initial manifold as time passes by, we may choose the parameter γ_3 to coincide with the time coordinate such that $\partial\mathbf{r}/\partial\gamma_3 = (0,0,1) = \mathbf{n}$, i.e., the unit vector in the direction of the time coordinate. Furthermore, the time coordinate can be considered, without loss of generality, to be perpendicular to the horizontal plane formed by the x and y coordinate, i.e., $c_o t$ represents the z-axis. In this way, $\mathbf{n}$ is the unit vector normal to the horizontal plane. The term $\partial\mathbf{x}/\partial\gamma_2$ represents a vector tangent to the action front from which a family of rays is spreading out. Thus the width between two adjacent rays, $d\nu$, can be expressed as [3]

$$d\nu = |\,\mathbf{n}\cdot(\mathbf{s}\times\frac{\partial\mathbf{x}}{\partial\gamma_2})d\gamma_2\,| = |\,Jd\gamma_2\,|\,. \qquad (26)$$

Here $d\gamma_2$ is the infinitesimal interval of length identifying two adjacent rays with parameter values γ_2 and $\gamma_2+d\gamma_2$ along the initial manifold. Since $d\gamma_2$ is a constant, it follows from Eqs. (22) and (6) that

$$\frac{\delta E}{\sigma}d\nu\,|\,\mathbf{C_g}+\mathbf{U}\,|\left[1+\frac{c_o^2}{|\mathbf{C_g}+\mathbf{U}|^2}\right]^{1/2} = constant. \qquad (27)$$

[3] A triple scalar product $\mathbf{n}\cdot(\mathbf{A}\times\mathbf{B})$ represents an area of the orthogonal projection of the parallelogram determined by vectors $\mathbf{A}$ and $\mathbf{B}$ onto a plane whose unit normal is $\mathbf{n}$.

The Unsteadiness Factor

The factor $c_o/(\mathbf{C_g} + \mathbf{U}) \equiv \Psi$ in (27) can be considered as a contribution to the wave height amplification due to the unsteadiness of the ambient medium. This factor is equivalent to the one suggested by Tolman(1990) as a measure of the unsteadiness of the depth or the current field. The major cause of unsteadiness in water depth and current in the coastal region is tides and tidal currents. The value c_o in this situation represents the phase speed of the tide in the direction of wave propagation, i.e., $c_o \equiv C_t \cos\alpha \sim \sqrt{gh}\cos\alpha$, where α is the angle between the tide and the direction of wave propagation . Since $\sqrt{gh} > \mid \mathbf{C_g} + \mathbf{U} \mid$, Ψ is in the order of one or larger.

To provide a rough idea about the magnitude of Ψ, we consider a situation where waves propagate in water of constant depth, say $h = 25,m$ following the direction of tide. We take the amplitude of tides to be $A' = 0.5\ m$ in the open ocean where the depth is typically $h' = 4000\ m$, then $c_o' = 198\ m/s$ and $U' = c_o'A'/h' = 2.5\ cm/s$ (Bowden, 1983). The corresponding values in water of $h = 25\ m$ are $A = A'(h/h')^{1/4} = 1.8\ m$, $c_o = c_o'(h/h')^{1/2} = 15.7\ m/s$, $U = U'(h'/h)^{3/4} = 1.1\ m/s$. For waves of period 15 seconds propagated in the water of 25 meter depths, the group velocity is $C_g = 12.4\ m/s$, we have $\Psi = 1.2$. For waves of period 10 seconds, $C_g = 9.4\ m/s$, $\Psi = 1.5$ and for 5 second waves, $C_g = 3.9\ m/s$, $\Psi = 3.1$. Thus, the effect of unsteadiness in the ambient medium on short waves can be substantial.

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