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form. Various examples of the use of these models will be shown in the conference presentation.

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ON THE TRANSFORMATION OF WAVE SPECTRA BY CURRENTS AND BATHYMETRY¹

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Abstract

Application of the conservation principle for action spectral density along rays, frequently described in literature, is not sufficient to specify the refracted wave spectrum. In order to determine the change of wave spectrum by current-depth refraction correctly, the effect of ray separation which causes convergence or divergence of wave energy must be included. This effect can be derived from the divergence of the velocity field in the conservation equation for action spectral energy density.

Introduction

A number of papers which apply ray theory to calculate refracted wave spectra have appeared recently (e.g., Le Méhauté and Wang, 1982; Mathiesen, 1987; Liu et al., 1989). However, the theoretical bases upon which these calculations are made are not always clear. Some apply only the conservation principle for action spectral density while others include an additional transformation factor based on purely mathematical reasoning. The purpose here is to clarify the problem from a theoretical point of view and to provide physical insight to this transformation factor.

Basic Equations

The change of wave field due to the presence of varying currents and bathymetry can be specified based on conservation of wave action along characteristic curves or rays (Bretherton and Garrett, 1969; Phillips, 1977). The path of a ray is determined by simultaneous solution of the following set of equations:

$$\frac{dk_i}{dt} = -\frac{\partial\omega}{\partial\lambda} \frac{\partial\lambda}{\partial x_i} = -\frac{\partial\sigma}{\partial h} \frac{\partial h}{\partial x_i} - k_j \frac{\partial u_j}{\partial x_i}, \quad (1)$$

$$\frac{dx_j}{dt} = \frac{\partial\omega}{\partial k_j} = c_{g_j} + u_j, \quad (2)$$

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$$\frac{d\omega}{dt} = \frac{\partial\omega}{\partial\lambda} \frac{\partial\lambda}{\partial t} = \frac{\partial\sigma}{\partial h} \frac{\partial h}{\partial t} + k_i \frac{\partial u_i}{\partial t}, \quad (3)$$

where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \frac{\partial\omega}{\partial k_j} \frac{\partial}{\partial x_j}. \quad (4)$$

Tensor notation is used to make these relations more concise. Here we express the wave-number vector by $\mathbf{k} = (k_1, k_2)$, group velocity by $\mathbf{C}_g = (c_{g1}, c_{g2})$, flow velocity by $\mathbf{U} = (u_1, u_2)$, and the horizontal cartesian coordinates by $\mathbf{x} = (x_1, x_2)$. $\omega(\mathbf{k}, \lambda) = \sigma + k_i u_i$ represents the apparent frequency and σ , the intrinsic frequency of waves in a frame of reference moving with flow velocity $\mathbf{U}(\mathbf{x}, t)$, obeys the dispersion relationship, $\sigma = (gk \tanh kh)^{1/2}$, where g is the gravitational acceleration, $k = |\mathbf{k}|$, and $h(\mathbf{x}, t)$ is water depth. $\lambda(\mathbf{x}, t)$ represents local properties of the medium, i.e., h and \mathbf{U} . Equation 3 indicates that if water depth and current velocity do not vary with time, ω remains constant along the rays.

Conservation of wave action for a slowly varying wavetrain of small amplitude can be expressed in terms of rays as

$$\frac{d}{dt} \left(\frac{E}{\sigma} \right) + (\nabla \cdot \mathbf{V}) \left(\frac{E}{\sigma} \right) = 0. \quad (5)$$

E is the local wave energy per unit area (proportional to the square of the wave amplitude) and $\mathbf{V} = \mathbf{C}_g + \mathbf{U}$. The wave action is defined as E/σ . For a continuous spectrum, E corresponds to the energy density of a group of waves whose wave-numbers lie in the element of area δA of the wave-number plane, specified by the vectors \mathbf{k} , $\mathbf{k} + \delta\mathbf{k}'$, and $\mathbf{k} + \delta\mathbf{k}''$ such that

$$\delta E(\mathbf{k}) = \rho g F(\mathbf{k}) \delta A, \quad (6)$$

$$\delta A = |\delta\mathbf{k}' \times \delta\mathbf{k}''|. \quad (7)$$

$F(\mathbf{k})$ is the spectral density and ρ the water density. By applying the kinematic conservation principle, Phillips(1977) has shown that

$$\frac{d}{dt} \delta A + (\nabla \cdot \mathbf{V}) \delta A = 0. \quad (8)$$

Therefore

$$\frac{d}{dt} \left(\frac{F(\mathbf{k})}{\sigma} \right) = 0. \quad (9)$$

Equation 9 expresses the conservation of action spectral density along the ray. In the absence of a current, $F(\mathbf{k})$ remains constant along the ray. This result was first demonstrated by Longuet-Higgins(1957).

Ray Separation Factor

Equation 8 cannot be directly integrated along a ray because knowledge of neighboring solutions is required to determine the divergence of the velocity. In order to solve this problem, we introduce the Jacobian $J(s, r)$ of the transformation from the ray coordinates (s, r) to $\mathbf{x} = (x, y)$,

$$J(s, r) = \frac{\partial(x, y)}{\partial(s, r)}, \quad (10)$$

where r is a parameter which is a constant along each ray and s is the arclength along the ray. Then differentiating Eq. 10 with respect to s we obtain (Chao and Bertucci, 1989)

$$\frac{dJ}{ds} = \frac{d}{ds} \left\{ \frac{\partial x}{\partial s} \frac{\partial y}{\partial r} - \frac{\partial x}{\partial r} \frac{\partial y}{\partial s} \right\} = J \left[\mathbf{V}^{-1} \nabla \cdot \mathbf{V} + \mathbf{V} \cdot \nabla \mathbf{V}^{-1} \right]. \quad (11)$$

If the temporal variation of current and water depth is small compared to wave period

$$\frac{\partial}{\partial t} \mathbf{V}^{-1} = 0, \quad (12)$$

and we have from Eqs. 2 and 4 that

$$\mathbf{V} \cdot \nabla \mathbf{V}^{-1} = \frac{d}{dt} \mathbf{V}^{-1} = \mathbf{V} \frac{d}{ds} \mathbf{V}^{-1}. \quad (13)$$

By substituting Eq. 13 into Eq. 11 and rearranging, we obtain

$$\nabla \cdot \mathbf{V} = \mathbf{V} \left[\frac{1}{J} \frac{dJ}{ds} - \mathbf{V} \frac{d}{ds} \mathbf{V}^{-1} \right]. \quad (14)$$

Therefore Eq. 8 becomes

$$\frac{d}{dt} \ln(J \mathbf{V} \delta A) = 0, \quad (15)$$

which states that the quantity $J \mathbf{V} \delta A$ is conserved along the ray.

The Jacobian $J(s, r)$ can be given a geometric interpretation. We note that the Jacobian defined by Eq. 10 can also be expressed as

$$J(s, r) = (0, 0, 1) \cdot \frac{\partial \mathbf{x}}{\partial s} \times \frac{\partial \mathbf{x}}{\partial r}, \quad (16)$$

where $\partial \mathbf{x} / \partial s = \hat{s}$ is the unit tangent vector along a ray. Thus

$$\delta x \equiv J(s, r) \delta r \quad (17)$$

is the width between two neighboring rays associated with parameters r and $r + \delta r$. Since δr is a constant, we obtain from Eqs. 9, 15, and 17

$$F(\mathbf{k}) \delta A = \frac{\sigma}{\sigma'} \left\{ \frac{\delta x'}{\delta x} \frac{\mathbf{V}'}{\mathbf{V}} \right\} F'(\mathbf{k}') \delta A' = \frac{\sigma}{\sigma'} \left\{ \frac{\delta A}{\delta A'} \right\} F'(\mathbf{k}') \delta A'. \quad (18)$$